5TH YEAR HONOURS MATHEMATICS
SUM AND PRODUCT OF THE ROOTS OF QUADRATIC EQUATION

The general formula of a quadratic equation is written as $ax^2 + bx + c = 0$.
To make the coefficient of x^2 equal to 1, simply divide every term by a.
$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Since the highest power of a quadratic is 2, we expect the equation to have two roots. Therefore the roots of this equation may be represented by α and β. We have already studied how to form a quadratic equation from the known roots, so use the above roots to form the quadratic equation they come from.

The roots are α and β. $\Rightarrow x = \alpha$ or $x = \beta$. $\Rightarrow x - \alpha = 0$ or $x - \beta = 0$.

Since these terms are both = 0, when multiplied by one another the equation formed will also = 0.

$\Rightarrow (x - \alpha)(x - \beta) = 0$. Multiply out these brackets to get $x^2 - \beta x - \alpha x + \alpha\beta = 0$
Since -x is common to the two middle terms take it out $\Rightarrow x^2 - x(\alpha + \beta) + \alpha\beta = 0$

Again this is the general formula for a quadratic equations whose roots are α and β.

Set this formula equal to the one above $\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - x(\alpha + \beta) + \alpha\beta$.

Comparison of similar terms shows that $\frac{b}{a} = -(\alpha + \beta) \Rightarrow -\frac{b}{a} = (\alpha + \beta)$. Also $\frac{c}{a} = \alpha\beta$

In any quadratic equation, the coefficient of x will be = –(the sum of the roots) and the constant term at the end of the quadratic will be (the product of the roots).

The general form of a quadratic equation: $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

This identity allows us to form a new quadratic if given its roots or if given the roots in terms of α and β.

EXAMPLE:
The roots of the equation $2x^2 + 6x + 3 = 0$ are α and β. Find the value of
(i) $\alpha + \beta$ (ii) $\alpha\beta$ (iii) $3\alpha + 3\beta$ (iv) $\alpha^2 + \beta^2$

The roots of the quadratic equation $2x^2 + px + q = 0$ are $2\alpha + \beta$ and $2\beta + \alpha$.
Find the value of p and the value of q.

SOLUTIONS: ***(It is most important that we find $\alpha + \beta$ and $\alpha\beta$ as soon as possible)***

(i) We know that $-\frac{b}{a} = (\alpha + \beta) \Rightarrow (\alpha + \beta) = -\frac{6}{2} \Rightarrow (\alpha + \beta) = -3$

(ii) $\alpha\beta = \frac{c}{a} = \frac{3}{2}$

(iii) $3\alpha + 3\beta = 3(\alpha + \beta)$. We know that $(\alpha + \beta) = -3$ so $3(\alpha + \beta) = 3(-3) = -9$.

(iv) $\alpha^2 + \beta^2$ To get the terms α^2 and β^2 we must multiply $(\alpha + \beta)(\alpha + \beta)$.

$\Rightarrow (\alpha + \beta)(\alpha + \beta) = \alpha^2 + 2\alpha\beta + \beta^2$

(we know, from earlier work, that $\alpha + \beta = -3$ *and that* $\alpha\beta = \frac{3}{2}$*)*

$\Rightarrow (-3)(-3) = \alpha^2 + \beta^2 + 2\left(\frac{3}{2}\right) \Rightarrow 6 = \alpha^2 + \beta^2$

In the second part of this question, we are given the roots in terms of α and β. They are $2\alpha + \beta$ and $2\beta + \alpha$. We know that the general form of a quadratic equation is $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$, so we need to find the product of the roots and the sum of the roots.

Sum of roots = $2\alpha + \beta + 2\beta + a = 3\alpha + 3\beta = 3(\alpha + \beta)$

Product of roots = $(2\alpha + \beta)(2\beta + \alpha) = 4\alpha\beta + 2\alpha^2 + 2\beta^2 + \alpha\beta = 5\alpha\beta + 2(\alpha^2 + \beta^2)$

Substitute these into the general formula to get $x^2 - (3(\alpha + \beta))x + 5\alpha\beta + 2(\alpha^2 + \beta^2) = 0$

From part (i) – (iv) above, we calculated the value of $\alpha + \beta$ to be -3, the value of $\alpha\beta$ to be $\frac{3}{2}$ and the value of $\alpha^2 + \beta^2$ to be 6. We now substitute these values into the above equation to get: $x^2 - (3(-3))x + 5(\frac{3}{2}) + 2(6) = 0$

$\Rightarrow x^2 + 9x + \frac{15}{2} + 12 = 0 \Rightarrow x^2 + 9x + 19\frac{1}{2} = 0$

We need to find the value of p and q in $2x^2 + px + q = 0$. Therefore we must multiply the final equation by 2 $\Rightarrow 2x^2 + 18x + 39 = 0$. Hence p = 18 and q = 39.