As already established, the square of any real number is always positive:

$$\Rightarrow (x)^2 \geq 0,\ -\infty \leq x \leq \infty.$$

Therefore in algebra, the square of any algebraic term is also positive:

$$(a+b)^2 \geq 0 \text{ and } (a-b)^2 \geq 0.$$

We make use of this reality in order to prove most inequalities. The inequality to be proved is simply factorised, manipulated or substituted into until a perfect square, like those above, is formed on one side. We may also use results of previous parts of a question to build up an inequality we have been asked to prove.

In all there are three methods utilised in order to help prove an inequality:

(a) By forming a quadratic that has two identical factors leading to $(a+b)^2 \geq 0$ or $(a-b)^2 \geq 0$

(b) By using statements of fact that are given at the start of the question. These must be incorporated into the proof

(c) By starting with the fact that $(a-b)^2 \geq 0$ and building up the required proof

EXAMPLE:
Let a, b and c be positive unequal real numbers.
Using results $a^2+b^2>2ab$, $b^2+c^2>2bc$ and $c^2+a^2>2ac$,

(i) Deduce that $a^2-ab+b^2>ab$

(ii) Deduce that $a^2+b^2+c^2>bc+ca+ab$

(iii) Show that $a^3+b^3>ab(a+b)$

(iv) Show that $3(a^3+b^3+c^3)>(a^2+b^2+c^2)(a+b+c)$

SOLUTIONS: *Parts (i) and (iii) are proved using method (a) whilst parts (ii) and (iv) are proved using method (b).*

(i) $a^2-ab+b^2>ab \Rightarrow a^2-2ab+b^2>0 \Rightarrow (a-b)(a-b)>0 \Rightarrow (a-b)^2>0$ ✓

(ii) $a^2+b^2+c^2>bc+ca+ab$ Since we need to get a^2, b^2 and c^2 together on the same side, we use the results given in the question above as follows:

$a^2+b^2>2ab$
$b^2+c^2>2bc$
$c^2+a^2>2ac$

$2a^2+2b^2+2c^2>2ab+2bc+2ac \Rightarrow$ *(Now ÷ 2)* $\Rightarrow a^2+b^2+c^2>ab+bc+ac$ ✓

(iii) $a^3 + b^3 > ab(a+b)$

To begin, factorise the $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$\Rightarrow (a+b)(a^2 - ab + b^2) > ab(a+b)$

Now cancel the $(a+b)$ *from both sides and manipulate!*

$\Rightarrow a^2 - ab + b^2 > ab$

$\Rightarrow a^2 - 2ab + b^2 > 0$

$\Rightarrow (a-b)(a-b) > 0$

$\Rightarrow (a-b)^2 > 0$ ✓

(iv) $3(a^3 + b^3 + c^3) > (a^2 + b^2 + c^2)(a+b+c)$

Since we now need $a^3 + b^3 + c^3$ together on the same side we use the $a^3 + b^3 > ab(a+b)$ from the previous part and write out similar equations for $a^3 + c^3$ and $b^3 + c^3$ as follows:

$a^3 + c^3 > ac(a+c)$ and $b^3 + c^3 > bc(b+c)$

Now add these together:

$a^3 + b^3 > ab(a+b)$

$a^3 + c^3 > ac(a+c)$

$b^3 + c^3 > bc(b+c)$

$2a^3 + 2b^3 + 2c^3 > ab(a+b) + ac(a+c) + bc(b+c)$

Since the question requires $3(a^3 + b^3 + c^3)$, we add $a^3 + b^3 + c^3$ to both sides:

$\Rightarrow 3a^3 + 3b^3 + 3c^3 > a^3 + b^3 + c^3 + ab(a+b) + ac(a+c) + bc(b+c)$

Now multiply out all brackets and manipulate:

$\Rightarrow 3(a^3 + b^3 + c^3) > a^3 + a^2b + a^2c + b^3 + ab^2 + b^2c + c^3 + ac^2 + bc^2$

$\Rightarrow 3(a^3 + b^3 + c^3) > a^2(a+b+c) + b^2(a+b+c) + c^2(a+b+c)$

$\Rightarrow 3(a^3 + b^3 + c^3) > (a^2 + b^2 + c^2)(a+b+c)$ ✓