A linear equation has one root that will satisfy it. A quadratic has two roots that will satisfy it. A cubic expression has three roots that will satisfy it and so on. An inequality is an algebraic expression that has a range of values, usually between some critical values or boundary points that can satisfy it. This is because the = has been replaced by one of four signs >, <, ≤ or ≥ which makes both sides of the equation unequal to one another.

It is our exercise to find the boundary points of the inequality given and see if the solutions to the inequality lie inside or outside the boundary points.

Solving inequalities is exactly the same as solving the corresponding algebraic equation. The first thing to do with an inequality is replace the inequality sign with an =, then make sure one side is set to 0 and solve it in the usual manner.

There are three types of inequalities we must study: (i) quadratic, (ii) modular and (iii) rational (involving fractions).

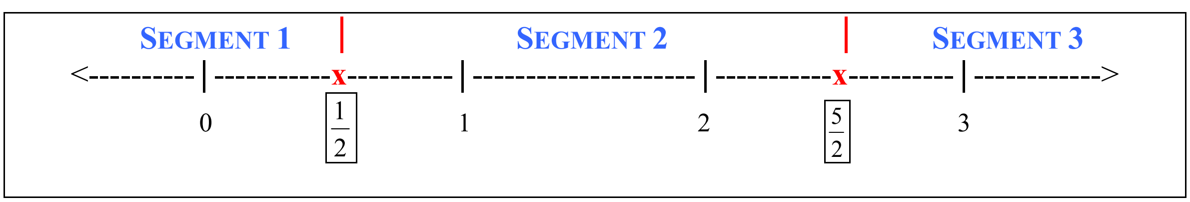
QUADRATIC INEQUALITIES

These are solved in the same way as quadratic equations. The solutions are the boundary points of the inequality.

EXAMPLE: Solve the following inequality for $x \in R$, $(2x-3)^2 \geq 4$.

SOLUTION: Expand $(2x-3)^2 \Rightarrow 4x^2 - 12x + 9$ and replace the $\geq$ with an =. $\Rightarrow 4x^2 - 12x + 9 = 4 \Rightarrow 4x^2 - 12x + 5 = 0$. Factorise this in the usual manner and find the roots of this quadratic: $\Rightarrow (2x-1)(2x-5) = 0 \Rightarrow x = \frac{1}{2}$ or $x = \frac{5}{2}$. These roots are the boundary points inside or outside of which the solutions to the inequality lie.

Plot these points on a number line and then take one value from each segment of the line and plug it into the original inequality:



- From segment 1 we take 0 $\Rightarrow (2(0)-3)^2 \geq 4 \Rightarrow 9 \geq 4$, which is true
- From segment 2 we take 2 $\Rightarrow (2(2)-3)^2 \geq 4 \Rightarrow 1 \geq 4$, which is false
- From segment 3 we take 3 $\Rightarrow (2(3)-3)^2 \geq 4 \Rightarrow 9 \geq 4$, which is true.

$\Rightarrow$ the solution is not between $\frac{1}{2}$ and $\frac{5}{2}$ $\therefore$ Solution is x < $\frac{1}{2}$ and x > $\frac{5}{2}$.

MODULAR INEQUALITIES

Whenever a modulus of a number appears in an equation, you should clear it by squaring all terms in that equation.

EXAMPLE: Solve the following inequality for $x \in R$: $2|x+1| \prec |x+3|$

SOLUTION: Square everything to remove the modulus signs from both sides and replace the < sign with an =.

$$\Rightarrow 4(x+1)^2 = (x+3)^2 \Rightarrow 4x^2 + 8x + 4 = x^2 + 6x + 9 \Rightarrow 3x^2 + 2x - 5 = 0$$

Now find the roots of this quadratic $(3x+5)(x-1) = 0 \Rightarrow x = -\frac{5}{3}$ or $x = 1$. These are the boundary points that must be plotted on the number line.

```
   SEGMENT 1   |               SEGMENT 2                  |  SEGMENT 3
<----------|------x--------------|---------------------|---------------------|------------>
          -2                    -1                     0                     1
```

Take a whole number value from each segment. Substitute it into the original inequality:

- From segment 1 take –2 $\Rightarrow 2|(-2)+1| \prec |(-2)+3|$ $\Rightarrow 2 < 1$
- From segment 2 take 0 $\Rightarrow 2|(0)+1| \prec |(0)+3|$ $\Rightarrow 2 < 3$ ✓
- From segment 3 take 2 $\Rightarrow 2|(2)+1| \prec |(2)+3|$ $\Rightarrow 6 < 5$

Since all values in the middle segment satisfy the original inequality the solution is between $-\frac{5}{3}$ and 1 $\Rightarrow$ $-\frac{5}{3} \prec x \prec 1$.

RATIONAL INEQUALITIES:

Be careful, it is really tempting to multiply both sides of the inequality by the denominator like you do when solving rational equations. The problem is the expression in the denominator will have a variable, so we won't know what the denominator is equal to. Remember that if we multiply both sides of an inequality by a positive number, it does not change the inequality. BUT if we multiply both sides by a negative, it does change the sign of the inequality. Since we don't know what sign we are dealing with we need to go about it the way described below.

STEP 1: Write the rational inequality in standard form. It is VERY important that one side of the inequality is 0.
STEP 2: Using a common denominator, simplify to a single fraction.
STEP 3: At this stage you may work with the equation on the top line of the fraction. *(You may if you wish also work with the denominator if it is easy to factorise.)* Simplify the top line, set it = 0 and solve to find the boundary points or the 'critical values' of the equation.

STEP 4: Plot these on a number line, define the segments, and plug a value from each segment back into the original inequality.
STEP 5: Those values that satisfy the original inequality reveal the segments that hold all the solutions to the inequality.

EXAMPLE: Solve $\left|\frac{x+2}{x-1}\right| \leq 3$

SOLUTION: Remove the modulus by squaring all terms: $\Rightarrow \frac{x^2+4x+4}{x^2-2x+1} \leq 9$

Set one side of the equation = 0: $\Rightarrow \frac{x^2+4x+4}{x^2-2x+1} - \frac{9}{1} \leq 0$.

Using a common denominator, simplify: $\Rightarrow \frac{x^2+4x+4-9(x^2-2x+1)}{x^2-2x+1} \leq 0$

It is necessary now only to work with the top line: $x^2+4x+4-9x^2+18x-9=0$

$\Rightarrow -8x^2+22x-5=0$ $\qquad \Rightarrow 8x^2-22x+5=0$ $\qquad \Rightarrow (4x-1)(2x-5)=0$

$\Rightarrow \boxed{x=\frac{1}{4} \text{ or } x=\frac{5}{2}}$

Now plot these boundary points on a number line:

```
   SEGMENT 1        |             SEGMENT 2                  |        SEGMENT 3
<---------|----x--------------|---------------------|------------x----------|--------->
          0                   1                     2                       3
```

Now test a value from each segment by plugging back into the original inequality:

When x = 0 $\Rightarrow \left|\frac{0+2}{0-1}\right| \leq 3 \quad \Rightarrow \left|\frac{2}{-1}\right| \leq 3 \quad \Rightarrow 2 \leq 3$ ✓

When x = 2 $\Rightarrow \left|\frac{2+2}{2-1}\right| \leq 3 \quad \Rightarrow \left|\frac{4}{1}\right| \leq 3 \quad \Rightarrow 4 \leq 3$ ✗

When x = 3 $\Rightarrow \left|\frac{3+2}{3-1}\right| \leq 3 \quad \Rightarrow \left|\frac{5}{2}\right| \leq 3 \quad \Rightarrow 2\frac{1}{2} \leq 3$ ✓

$\Rightarrow x \leq \frac{1}{4}$ and $x \geq \frac{5}{2}$