An index is used when a number is multiplied by itself over a number of times. Therefore, 5x5x5x5x5x5x5x5x5 can be written as 5^9. This makes the number easier to use. In 5^9, for example, 5 is called the ***base number*** and the ^{9} is the **index** or power. Numbers written in index form can then be multiplied, divided and raised to further powers. The main rule of indices is that **all numbers must be written in terms of their lowest base** before they can be used. If multiplication or division is involved, the base numbers must match before the multiplication or division can take place.

The rules concerning indices that you learned for the Junior Certificate are all you need for this section of the course! Some of the rules you learned are rewritten below in order to make the more difficult indices of this chapter easier to calculate. Lets recap:

$x^a.x^b = x^{a+b}$	$\frac{x^a}{x^b} = x^{a-b}$	$(x^a)^b = x^{ab}$	$(xy)^a = x^a.y^a$
$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$	$x^0 = 1$	$x^{-a} = \frac{1}{x^a}$	$x^{\frac{a}{b}} = \left(x^{\frac{1}{b}}\right)^a$

By far the most important one of these formulae is the last one, which will help us to solve questions involving fractional indices.

EXAMPLE: Evaluate $\sqrt{\frac{9^{\frac{1}{2}} - 2^{-1}}{32^{-\frac{3}{5}}}}$ and give you answer in the form $a\sqrt{b}$.

SOLUTION: Using the above rules, rewrite each number as follows:

$$= \sqrt{\frac{3 - \frac{1}{2}}{\left(\left(2^{\frac{1}{5}}\right)^5\right)^{-3}}} = \sqrt{\frac{5/2}{1/8}} = \sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

INDICES CONTAINING VARIABLES

In questions where we are asked to solve an index that has been written as a variable, we must always first rewrite each term to its lowest base and then, once all bases match, equate the indices and solve for the variable.

EXAMPLE: Find the value of n if $\frac{27^{\frac{2}{3}} \times 81^{-\frac{1}{2}}}{9^{-1}} = 3^n$

SOLUTION: Rewrite all terms to the base 3:

$27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2$; $\quad 81^{-\frac{1}{2}} = (3^4)^{-\frac{1}{2}} = 3^{-2}$; $\quad 9^{-1} = (3^2)^{-1} = 3^{-2}$

Now reform the fraction as $\frac{3^2 \times 3^{-2}}{3^{-2}} = 3^2$. Therefore, if $3^n = 3^2$, then $n = 2$.

Sometimes the whole expression will contain indices that are written in consecutive terms of n, e.g. $n-1$, n and $n+1$. In this case, not only have we to rewrite each number present to its lowest base, we must also rewrite each index so that it contains the lowest of the indices in the question. For example, if a question contains the indices $n-1$, n and $n+1$, you must rewrite all indices so that they contain the term $n-1$. Every term in the question will then contain a common base number with the lowest index, $n-1$. This makes it possible to divide across by that common term and remove it from the question altogether. This makes that question easier to evaluate.

EXAMPLE:
Evaluate the following $7.3^{2n+1} - 21.9^n$.

SOLUTION:

Step 1: Rewrite 9^n as $(3^2)^n = 3^{2n}$.

Step 2: Since $2n$ is the lowest index in the question, make sure that the index $2n$ is present in every other index. 3^{2n+1} can be rewritten as $3^{2n} \times 3^1 \Rightarrow 3^{2n}.3^1$. Rewriting the original equation gives $7.3^{2n}.3^1 - 21.3^{2n}$. Since each term has 3^{2n} as a common factor, we can divide each term by 3^{2n} in order to remove it. This leaves the equation as 7.3-21, which equates to 0.

If there are two variables present in the indices, then a pair of simultaneous equations must be formed so that we can solve for both variables.

EXAMPLE: Solve the following for x and y:

$$3^x.9^{-y} = 3$$
$$2^x.4^{2y} = 2$$

SOLUTION: Rewrite each term to its lowest base:

$$3^x.\left(3^2\right)^{-y} = 3^1$$
$$2^x.\left(2^2\right)^{2y} = 2^1$$

You may then ignore the bases numbers and equate the indices. This gives:

$$x - 2y = 1$$
$$x + 4y = 1$$

The simultaneous equations produced are solved in the usual manner.

Sometimes it is necessary to substitute y for a^x (any base with the index x) in order to produce an equation in y. This equation in y, is then easily factorised and solved. The solutions for y are then set equal to the original a^x and a solution for x is found.

EXAMPLE: Use the substitution $y = a^x$ to solve $2^{2x+1} - 3.2^x + 1 = 0$

SOLUTION: Here we need to rewrite 2^{2x+1} as $2^{2x}.2^1$

The 2^{2x} may be rewritten as $(2^x)^2$. This gives $(2^x)^2.2^1$ which, after substitution, becomes $2y^2$. The -3.2^x after substitution becomes $-3y$. The new equation now reads $2y^2 - 3y + 1 = 0$ which is a quadratic equation in y, and is solved in the usual manner.

In this case the roots are $y = \frac{1}{2}$ and $y = 1$.

Now replace y in these solutions with the 2^x to get $2^x = \frac{1}{2}$ and $2^x = 1$.

If $2^x = \frac{1}{2} \Rightarrow 2^x = 2^{-1} \Rightarrow x = -1$ and if $2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$.

RECURRENCE EQUATIONS

If you represent the n^{th} term in a sequence as $a[n]$, you can use a *recurrence equation* to specify how it is related to other terms in the sequence. In the case of 2, 4, 6, 8, etc the recurrence is +2 to the previous term, so $n + 2$ is the formula that will reveal any term. For example the 50th term in the above sequence is 50 + 2 = 52.

A recurrence equation, therefore is a formula which will generate any term in a sequence from previous terms and the notation used is known as sequence notation.

We will always be given the n^{th} term as u_n and must, by simple substitution for n, find out what u_{n+1} and u_{n+2} are.

EXAMPLE: If for all integers n, $u_n = 5.3^n + 4^n$, show that $u_{n+2} - 7u_{n+1} + 12u_n = 0$.

SOLUTION: Given u_n, we need to work out u_{n+1} and u_{n+2} since they are present in the final equation. We then substitute these back into the equation we are asked about.

$$u_n = 5.3^n + 4^n$$

$$\Rightarrow \quad u_{n+1} = 5.3^{n+1} + 4^{n+1} = 5.3^n.3^1 + 4^n.4^1 = 15.3^n + 4.4^n$$

$$\text{and} \quad u_{n+2} = 5.3^{n+2} + 4^{n+2} = 5.3^n.3^2 + 4^n.4^2 = 45.3^n + 16.4^n$$

Now substitute u_n, u_{n+1} and u_{n+2} into $u_{n+2} - 7u_{n+1} + 12u_n$.

$\Rightarrow \quad 45.3^n + 16.4^n - 7\left(15.3^n + 4.4^n\right) + 12\left(5.3^n + 4^n\right)$

$\Rightarrow \quad 45.3^n + 16.4^n - 105.3^n - 28.4^n + 60.3^n + 12.4^n$

$\Rightarrow \quad 0$

Since the 3^n and the 4^n terms all cancel out, we have shown that $u_{n+2} - 7u_{n+1} + 12u_n = 0$ as asked.