THE FACTOR THEOREM

A **factor** is a number that divides into another number with no remainder. e.g. 3 is a factor of 9, 4 is a factor of 24, 13 is a factor of 39 and so on.

Likewise in algebra. A factor of an algebraic term divides into that algebraic term leaving no remainder. We already know how to factorise a quadratic equation by the bracket method. e.g. Factorise $6x^2 - 11x - 10 \Rightarrow (3x+2)(2x-5)$. This procedure reveals the two terms that will divide evenly into the original quadratic. Hence $3x+2$ and $2x-5$ are called 'factors' of the quadratic $6x^2 - 11x - 10$.

Now, in order to solve the above quadratic equation we need to know what the equation is equal to. In every case of solving quadratic equations the equation must be set = 0 before it can be solved. Solving quadratic equations reveals values for x that satisfy the equation: e.g. Solve $6x^2 - 11x - 10 = 0 \Rightarrow (3x+2)(2x-5) = 0 \Rightarrow 3x+2=0$ or $2x-5=0 \Rightarrow x = -\frac{2}{3}$ or $x = \frac{5}{2}$. If we plug either $-\frac{2}{3}$ or $\frac{5}{2}$ into the original quadratic for x, the equation is satisfied and turns to 0.

Values that satisfy an algebraic equation are called the **roots** of that equation.

Looking closely at the factors and roots of the above equation we can see that they are very closely related. The factors are $(3x+2)$ and $(2x-5)$ therefore the roots are $x = -\frac{2}{3}$ and $x = \frac{5}{2}$.

The factor theorem states that if $(x-k)$ is a factor of an algebraic equation, say $f(x)$, then a root of that equation will be $x = k$ and $f(k) = 0$.

Likewise if $(ax-b)$ is a factor of an algebraic equation, say $g(x)$, the a root of that equation will be $x = \frac{b}{a}$ and $g\left(\frac{b}{a}\right) = 0$

EXAMPLE $f(x) = 3x^2 - 14x + 8$

$\Rightarrow$ Factors of $f(x) = (3x+2)(x+4)$

(Now set factors = 0 and solve to get roots)

$\Rightarrow$ Roots are $x = -\frac{2}{3}$ or $x = -4$

$\Rightarrow f\left(-\frac{2}{3}\right) = 0$ and $f(-4) = 0$

Factors are intrinsically linked with division. When we know one factor of a number we simply divide it into that number to find the other factor: e.g. 6 is a factor of 24. What is the other factor? Well 24 ÷ 6 = 4. Simple!

Again this procedure is the same with Algebra. If we know one factor of an algebraic function, then by long division we can find the other factor(s).

EXAMPLE: Verify that $(2x-3)$ is a factor of $2x^3-15x^2+34x-24$ and find the other two factors.

SOLUTION: Since $(2x-3)$ is a factor we know that it divides into the polynomial with no remainder.
Therefore set this factor equal to 0 and manipulate it find a root of the equation:

$\Rightarrow 2x-3=0 \Rightarrow x=\frac{3}{2}$. Now substitute this value for x in the polynomial:

$$\Rightarrow 2\left(\frac{3}{2}\right)^3-15\left(\frac{3}{2}\right)^2+34\left(\frac{3}{2}\right)-24 \Rightarrow \frac{54}{8}-\frac{135}{4}+\frac{102}{2}-\frac{24}{1}.$$

Multiply by 8 to clear all fractions $\Rightarrow 54-270+408-192=0$

The equation goes to 0 proving that $(2x-3)$ is a factor of the polynomial.

To find the other two factors we must divide the given factor into the polynomial.

$$\begin{array}{r|l} & x^2-6x+8 \\ \hline 2x-3 & 2x^3-15x^2+34x-24 \\ & \mp 2x^3 \pm 3x^2 \\ \hline & -12x^2+34x \\ & \pm 12x^2 \mp 18x \\ \hline & 16x-24 \\ & \mp 16x \pm 24 \\ \hline & 0 \end{array}$$

The other two factors are in the quadratic equation that appears after division. It must be factorised to reveal the other two factors we are looking for:

$x^2-6x+8 \Rightarrow (x-4)(x-2)$. So the three factors are $(2x-3)(x-4)$ and $(x-2)$.

NOTE: If there is a term missing from a polynomial that has to be divided, then fill in the missing term with a coefficient of 0: e.g. $ax^3 + bx + c$ is missing an x^2 term so rewrite it as $ax^3 + 0x^2 + bx + c$.

EXAMPLE: $x^2 + bx + c$ is a factor of $x^3 - p$. Show that (i) $c = b^2$ (ii) $bc = p$.

Firstly rewrite the polynomial to be divided as $x^3 + 0x^2 + 0x - p$. Now divide it by $x^2 + bx + c$ as shown:

$$\begin{array}{r|l} & x - b \\ x^2 + bx + c & x^3 + 0x^2 + 0x - p \\ & \mp x^2 \mp bx^2 \mp cx \\ & \quad -bx^2 - cx - p \\ & \quad \pm bx^2 \pm b^2x \pm bc \\ & \qquad -cx + b^2x - p + bc \end{array}$$

It is clear to see that no further division can now take place. Therefore, the last term $-cx + b^2x - p + bc$ must be the remainder. And since we have been told that $x^2 + bx + c$ is a factor, this remainder is = 0.

$\Rightarrow -cx + b^2x - p + bc = 0 \Rightarrow -cx - p = -b^2x - bc$ Now compare similar terms:

$\Rightarrow -cx = -b^2x \Rightarrow c = b^2$

Likewise, $-p = -bc \Rightarrow p = bc$