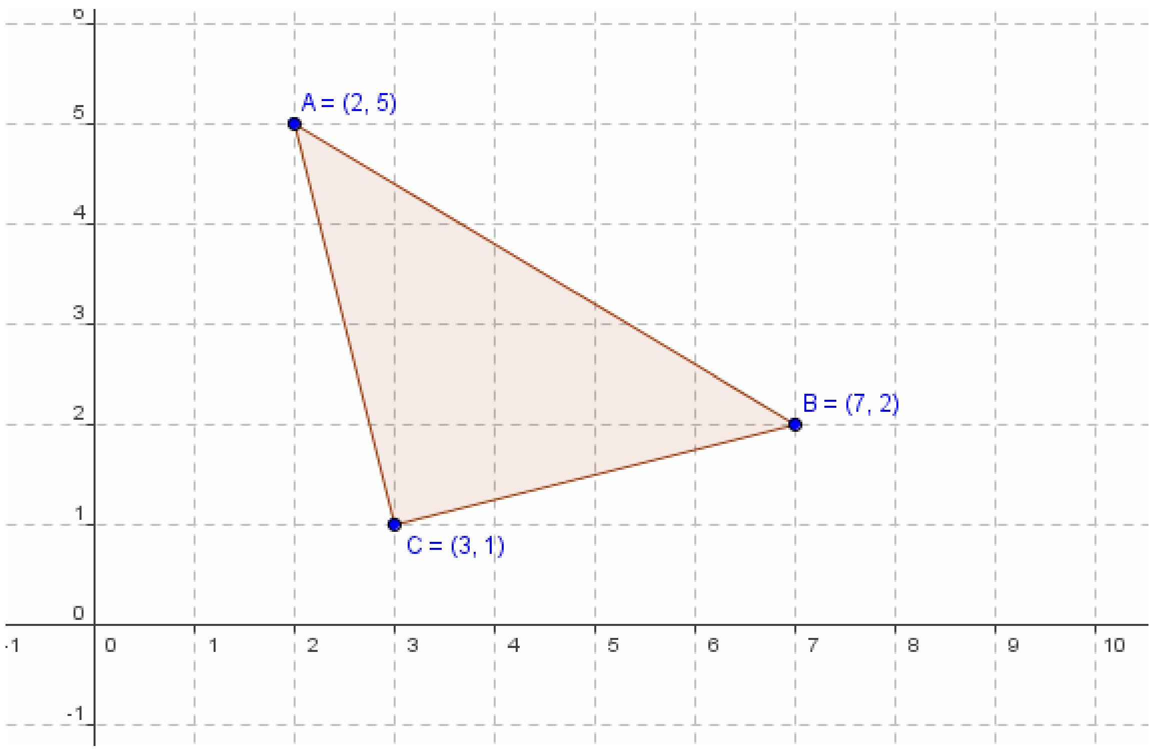
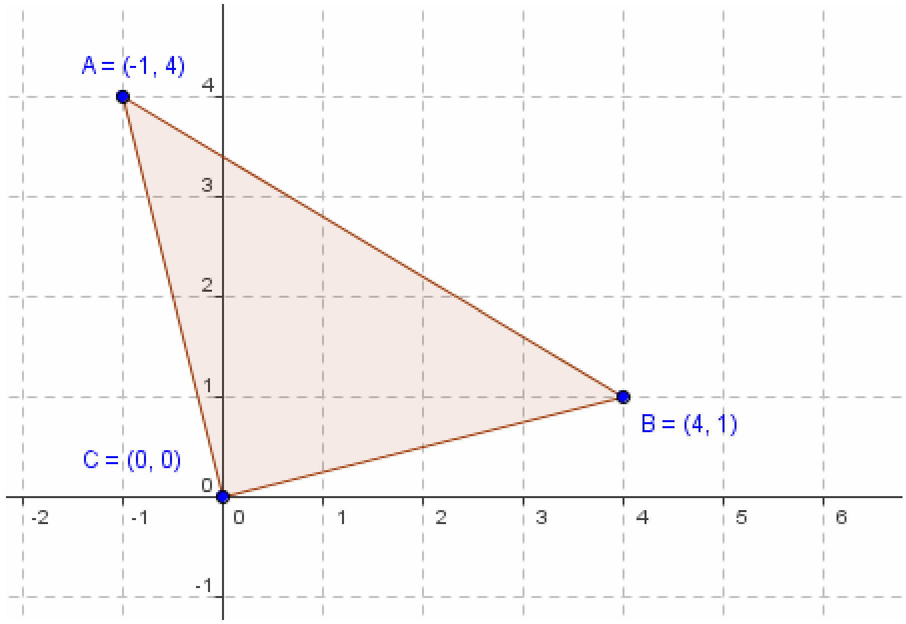
THE AREA OF A TRIANGLE

Following on from the coordinate geometry learned about the line for Junior Cert we look again at the area of a triangle. We have already learned 2 formulae through which we can find the area of a triangle. One is the basic $\frac{1}{2}b \times \perp h$ and the other, from trigonometry is $\frac{1}{2}ab\sin C$. As yet, we have no way of finding the area of a triangle that is drawn on the xy plane as follows:



The area formula used in this case is $\frac{1}{2}|x_1y_2 - x_2y_1|$. There is a slight problem in that the formula only allows for two coordinates (x_1, y_1) and (x_2, y_2) to be used. In a case like this one of the vertices, usually the closest, must be translated to the origin so that it has an x and y of 0. To ensure that the shape of the triangle stays the same, the other vertices must be adjusted to the same degree. This gives the same triangle in a different location with one of its vertices on (0,0). The two remaining vertices are used in the formula $\frac{1}{2}|x_1y_2 - x_2y_1|$.

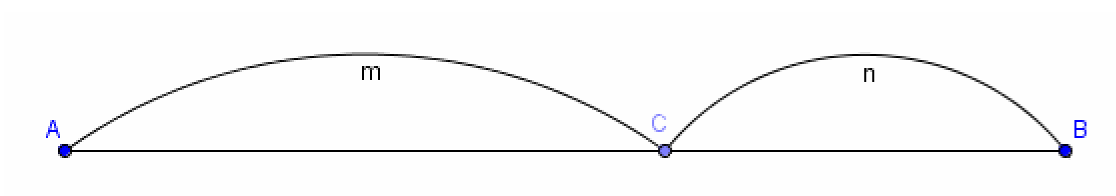
Since (3,1) → (0,0), (7,2) → (4,1) and (2,5) → (-1,4). The (0,0) is ignored and the points (4,1) and (-1,4) are used in the area formula.



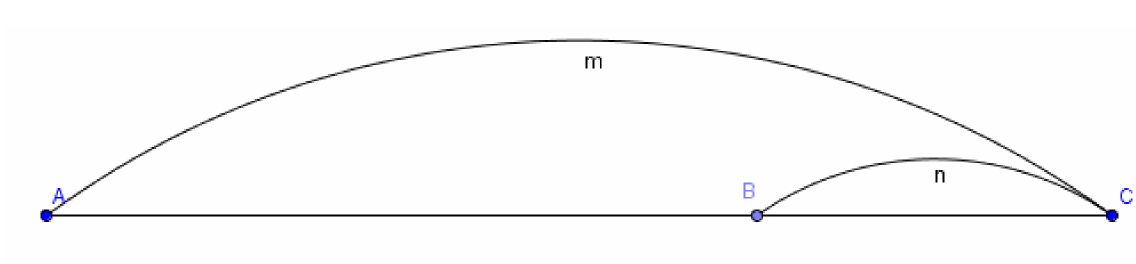
Here $\frac{1}{2}|x_1y_2 - x_2y_1|$ becomes $\frac{1}{2}|(-1)(1)-(4)(4)| = \frac{1}{2}|-17| = 8.5$ sq units. Notice why the modulus sign is so important. It is impossible to have –area, so the modulus guarantees a positive result.

DIVISION OF A LINE SEGMENT IN A GIVEN RATIO

A given line segment [AB] may be divided up in one of two ways:
1. Internally where the original line is broken up into two smaller lines by another point C



2. Externally where the original line is extended to another point C



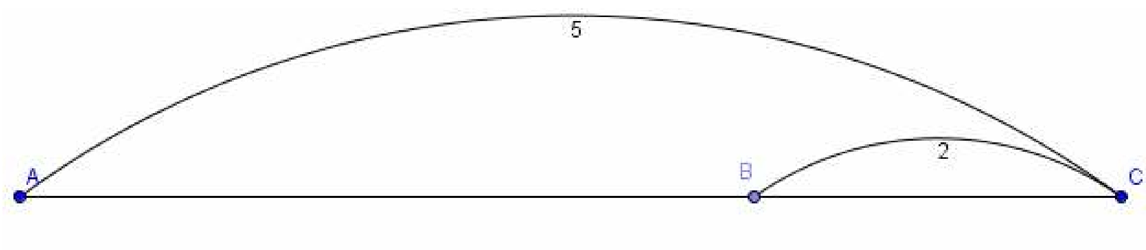
In either case it is clear to see that the line is broken up according to a definite ratio, defined by the letters m and n.

The coordinates of the dividing point C can be found using one of the following formulae

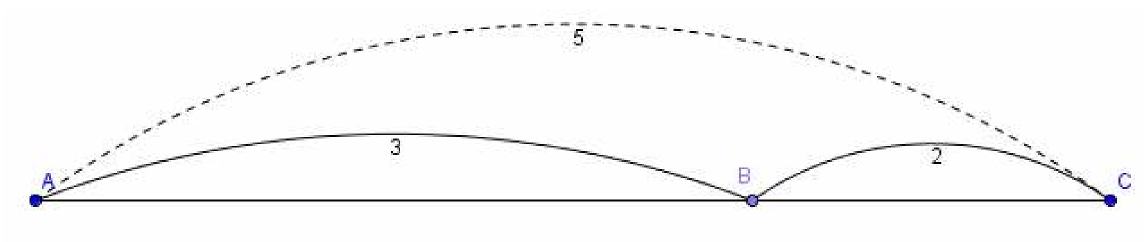
INTERNAL DIVISOR	**EXTERNAL DIVISOR**
$C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$	$C = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right)$

(m and n are the values of the dividing ratio, and x_1, y_1, x_2, y_2 *are the coordinates of the end points of the line AB)*

The case of the external division, can be made a little easier. The ratio may be written as an **external division of *AB*** or as an **internal division of *AC***. Take the case of this external division where $|AC|:|BC|$ = 5:2



This gives m = 5 and n = 2. We can use this external ratio to calculate the internal ratio of the line ac: If $|AC|:|BC|$ = 5:2, then $|AB|:|BC|$ = 3:2.



Recognising this we can apply this ratio to the points A, B and C and then apply the ratio to the x values and the y values of the three points on the line. This is a similar method to the way we solved similar triangles in Junior Cert Maths.

Example:

$a(3,2)$ and $b(18,12)$ are two points. $[ab]$ is produced to to c such that $|ac|:|bc| = 7:2$
Find the coordinates of the point c.

METHOD 1 - BY FORMULA	**METHOD 2 - BY SIMILAR TRIANGLES**
$c=\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)$ m = 7, n = 2, $x_1 = 3$, $y_1 = 2$, $x_2 = 18$ and $y_2 = 12$. This gives: $c=\left(\frac{7.18-2.3}{7-2},\frac{7.12-2.2}{7-2}\right)$ $c=(24,16)$	If the external ratio $\lvert ac\rvert:\lvert bc\rvert$ is 7:2, then the internal ratio $\lvert ab\rvert:\lvert bc\rvert$ is 5:2 Now compare the x values across the line and apply the internal ratio on [ac] **5 : 2** x's: $3 \rightarrow 18 \rightarrow 24$ y's: $2 \rightarrow 12 \rightarrow 16$ $\Rightarrow c = (24,16)$

CONCURRENCIES OF A TRIANGLE:
(Animations of their constructions are available in the 5th year section of the school's maths website at http://www.knocklyoncs.ie/maths)

Centroid: The point of intersection of at least two medians of a triangle. Since the point of intersection cuts all the medians into a ratio of 2:1, its coordinates can be found by getting ⅓ of the sum of the x values of all the vertices of the triangle and ⅓ of the sum of the y values of all the vertices.

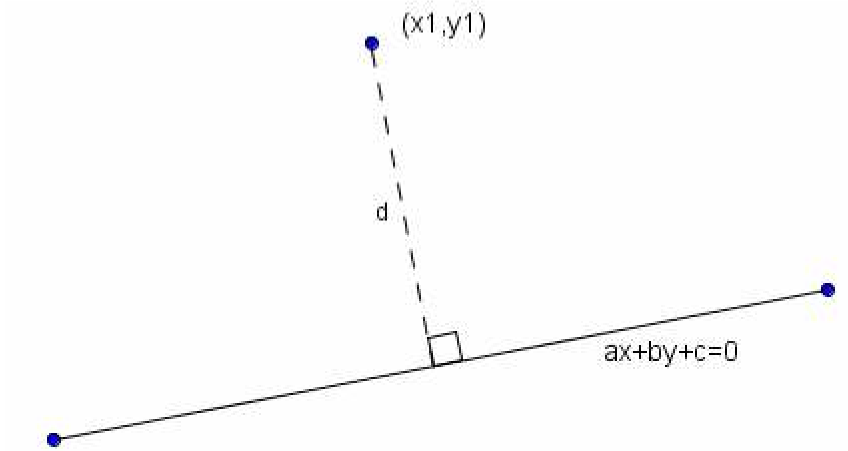
Coordinates of the centroid of a $\Delta = \left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

Circumcentre: The point of intersection of at least two perpendicular bisectors of the sides of a triangle. Found by applying coordinate geometry and simultaneous equations.

Orthocentre: The point of intersection of at least two lines from vertex perpendicular to the opposite side. Found by applying coordinate geometry and simultaneous equations.

The centroid, circumcentre and orthocenter of a given Δ all lie on a line called ***'Euler's Line'***.

PERPENDICULAR DISTANCE FROM A POINT TO A LINE



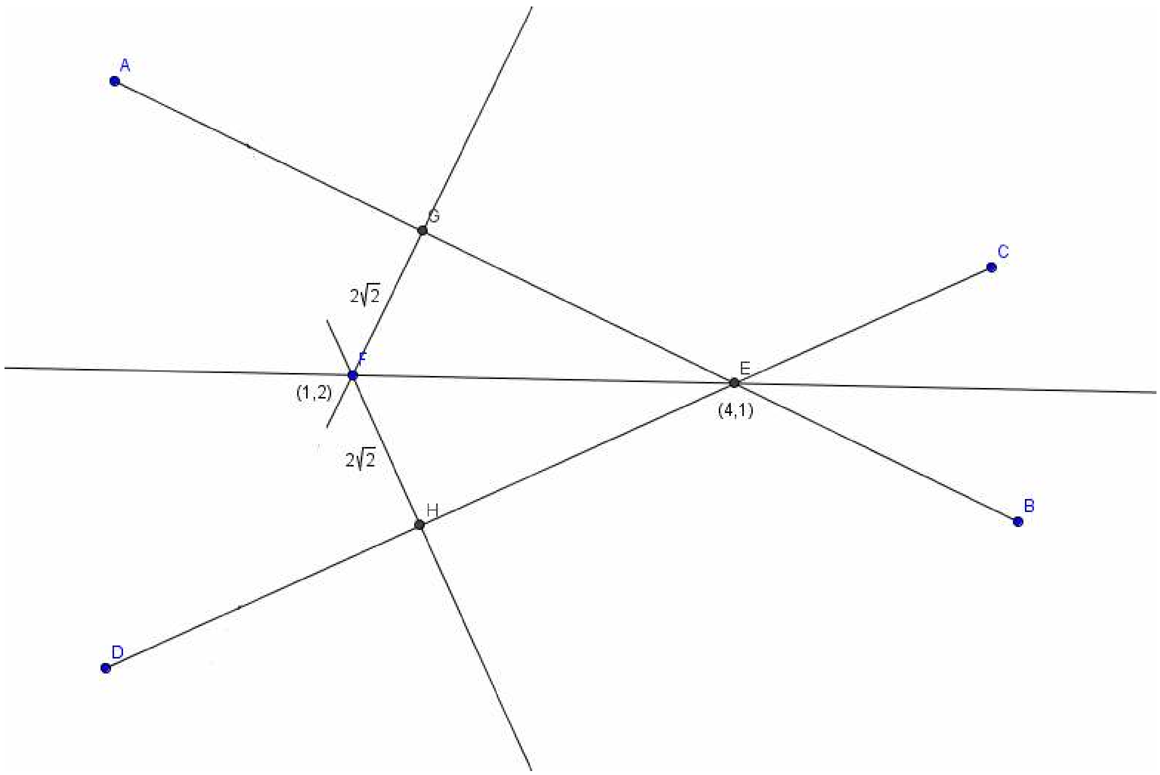
The perpendicular distance, d, from a point (x_1, y_1) to a line $ax+by+c=0$ is given by the following formula: $d = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.

- ***Locus***: A line along which a point can slide and remain equidistant from two other points.
- If line A = $ax+by+c=0$, then the equation of any other line $||$ to A is $ax+by+k=0$.
- If line A = $ax+by+c=0$, then the equation of any other line $\perp$ to A is $-bx+ay+k=0$.
- The slope of a line can be found by getting the tan of the angle it makes with the + side of the x-axis.

If we are asked to get the distance between two parallel lines then we must find one point on one of the lines and use the formula above with that point and the equation of the other line.

Example:
Find the equations of two lines which contain the point (4,1) and are a distance of $2\sqrt{2}$ units from (1,2)

Since we are asked for the equation of the two lines through (4,1), that are both a $\perp$ distance of $2\sqrt{2}$ units from (1,2) we construct the following diagram to see the question:



We are asked for the equation of the two lines $|AB|$ and $|CD|$. We need, therefore, a point on these lines which is (4,1), their slopes, *m*, and the formula for finding the equations of a line which is $y - y_1 = m(x - x_1)$.

Since we have no way of knowing the slopes of the two lines, we simply use the symbol *m* in the formula to get the ***general form*** of the equation of these lines:

$$y - 1 = m(x - 4) \Rightarrow y - 1 = mx - 4m \Rightarrow mx - y - 4m + 1 = 0$$

$mx - y - 4m + 1 = 0$ is the general form of the equation of the two lines we are looking for and is dependent only on the value of *m*. The next step, therefore, is to find the ***values*** of *m*. Since we are looking for 2 values of m, one for each line, we can expect to be solving a quadratic equation in *m*.

We have been given the fact that both lines are a ⊥ distance of $2\sqrt{2}$ from the point (1,2) so we use the ⊥ distance formula and set it equal to $2\sqrt{2}$.

$$\Rightarrow \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = 2\sqrt{2} \qquad \text{where} \qquad \begin{aligned} x_1 &= 1 \\ y_1 &= 2 \\ a &= m \\ b &= -1 \\ c &= -4m + 1 \end{aligned}$$

$\Rightarrow \dfrac{|(m)(1)+(-1)(2)+(-4m+1)|}{\sqrt{m^2+1}} = 2\sqrt{2} \quad \Rightarrow \dfrac{|m-2-4m+1|}{\sqrt{m^2+1}} = 2\sqrt{2} \quad \Rightarrow \dfrac{|-3m-1|}{\sqrt{m^2+1}} = 2\sqrt{2}$

Now square all terms to clear the modulus $\Rightarrow \dfrac{9m^2+6m+1}{m^2+1} = 8$

Cross multiply $\Rightarrow 8m^2+8 = 9m^2+6m+1 \Rightarrow m^2+6m-7=0$

Factorise and solve this quadratic:

$$\Rightarrow m^2+6m-7=0$$
$$\Rightarrow (m-1)(m+7)=0$$
$$\Rightarrow m=1 \text{ and } m=-7$$

These values of m are filled into the general equation, $mx-y-4m+1=0$, to give:

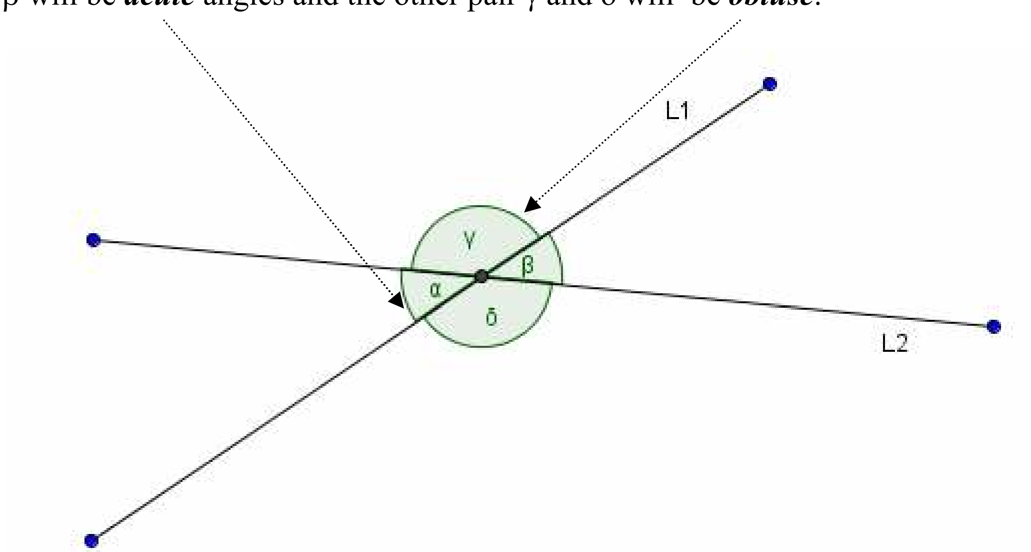
When $m=1$
$\Rightarrow x-y-3=0$

and when $m=-7$
$\Rightarrow -7x-y+29=0$
$\Rightarrow 7x+y-29=0$

The two equations are $x-y-3=0$ and $7x+y-29=0$

ANGLE BETWEEN TWO LINES

When two lines have different slopes, they will intersect at some point (x,y).
When they intersect, they will form two pairs of vertically opposite angles. One pair α and β will be ***acute*** angles and the other pair γ and δ will be ***obtuse***.



For this part of the course we will focus on the acute angle formed at the intersection.

It has been found that $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$. If we need to find the obtuse angle, simply take the acute one away from 180°.

Example:
Find the equations of the lines through the point (4,3) which make an angle of 45° with the line $6x + y - 5 = 0$.

Remember: To get the equation of any line we need a point (x,y) a slope, m, and the formula $y - y_1 = m(x - x_1)$.

In this question we have the point (4,3) which is common to both lines, and the slope of $6x + y - 5 = 0$ which is -6. We do not know the slopes of the lines passing through (4,3) so we use the variable *m* in their place. Since we are now looking for two slopes we ultimately must form a quadratic equation in *m*.

The exercise is now one of finding two values of *m*.

$$\tan 45 = \left|\frac{-6 - m}{1 - 6m}\right| \Rightarrow 1 = \left|\frac{-6 - m}{1 - 6m}\right|$$

Now square both sides to clear the modulus $\Rightarrow 1 = \frac{36 + 12m + m^2}{1 - 12m + 36m^2}$

Cross multiply $\Rightarrow 36m^2 - 12m + 1 = m^2 + 12m + 36$
$\Rightarrow 35m^2 - 24m - 35 = 0$
$\Rightarrow (7m + 5)(5m - 7) = 0$
$\Rightarrow m = -\frac{5}{7}$ or $m = \frac{7}{5}$

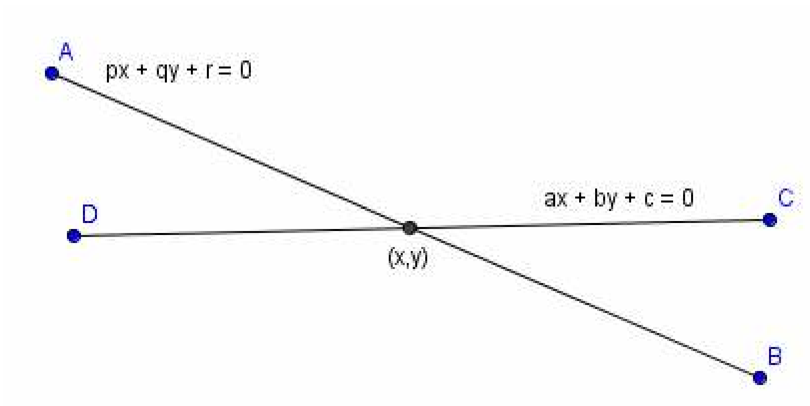
These slopes are now used individually with the point (4,3) and the formula $y - y_1 = m(x - x_1)$ to get the two equations of the lines making 45° with $6x + y - 5 = 0$.

When $m = -\frac{5}{7} \Rightarrow y - 3 = -\frac{5}{7}(x - 4) \Rightarrow 5x + 7y - 41 = 0$

When $m = \frac{7}{5} \Rightarrow y - 3 = \frac{7}{5}(x - 4) \Rightarrow 7x - 5y - 13 = 0$

CONCURRENT LINES

When two lines intersect, they meet at a single point that both lines hold in common. In the case of $ax+by+c=0 \cap px+qy+r=0$, shown below, the point is (x,y).



It is also true to say that there are an infinite number of lines that can pass through (x,y).

Since these other lines contain the point of intersection (x,y), they must be related to $ax+by+c=0$ and $px+qy+r=0$, and share a general formula derived from the two lines. The only thing that differentiates one line passing through (x,y) from another is the slope of that line.

The general formula of any line passing through the point of intersection of $ax+by+c=0$ and $px+qy+r=0$ is given as $\mu(ax+by+c)+\lambda(px+qy+r)=0$, where μ and λ are variables that, when varied, will give the equations of lines that pass through (x,y) with different slopes.

When given one piece of information in a question we let $\mu=1$ and consider $(ax+by+c)+\lambda(px+qy+r)=0$.

Example:
Find the relationship between the parameters μ and λ, where $\mu, \lambda \neq 0$, for which the line $\mu(3x-2y+3)+\lambda(x-2y-5)=0$

(i) makes and angle measuring 45° with the positive sense of the x-axis.
(ii) Contains the origin (0,0)

Solution:

(i) The tan of the angle a line makes with the positive side of the x-axis will reveal the slope of that line:

Angle made by above line = 45°, $\Rightarrow$ tan 45° = 1 so the slope of the line is 1.

Now get $\mu(3x-2y+3)+\lambda(x-2y-5)=0$ into general line form, $ax+by+c=0$ by removing brackets and gathering like terms:

$\Rightarrow (3\mu+\lambda)x - y(2\mu+2\lambda)+3\mu-5\lambda=0$

Now $\frac{-a}{b}$ will reveal the slope of this line $\Rightarrow \frac{-(3\mu+\lambda)}{-(2\mu+2\lambda)} \Rightarrow \frac{(3\mu+\lambda)}{(2\mu+2\lambda)}$

We also know that the slope of this line is 1 $\Rightarrow \frac{(3\mu+\lambda)}{(2\mu+2\lambda)}=1$

Cross multiply $\Rightarrow 3\mu+\lambda=2\mu+2\lambda \Rightarrow \mu=\lambda$

(ii) If the line contains the origin, then substitute x = 0 and y = 0 into the equation: $\mu(3x-2y+3)+\lambda(x-2y-5)=0$

$\Rightarrow \mu(3)+\lambda(-5)=0$

$\Rightarrow 3\mu=5\lambda$

PARAMETRIC EQUATIONS OF A LINE

Until now the general form of the equation of a line has been written as $ax+by+c=0$. This is known as the Cartesian form of the equation of a line.

If we write x in terms of a third variable like t, and y in terms of the same variable, then these two equations are known as the parametric equations of a line and the variable, t, is said to be the parameter.

In order to write the parametric equations of x and y in the more recognizable Cartesian form, simply manipulate both parametric equations so that t is written in terms of x, and t is written in terms of y. This will give two values for t which we then equate and manipulate so that the x is placed first, the y second and the constant term third. Of course the equation is set = 0.

Example:
Find the obtuse angle between the two lines whose parametric equations are:

$x=t+2$, $y=3-3t$ and $x=\frac{2t-1}{t-1}$, $y=\frac{2t}{t-1}$

Solution:

Firstly we must work out the Cartesian form of each line:

$$x = t + 2 \;\Rightarrow\; x - 2 = t$$
$$y = 3 - 3t \;\Rightarrow\; 3t = 3 - y \;\Rightarrow\; t = \frac{3-y}{3}$$

Since $x - 2 = t$ and $t = \frac{3-y}{3}$, then $x - 2 = \frac{3-y}{3} \Rightarrow 3x - 6 = 3 - y \Rightarrow \boxed{3x + y - 9 = 0}$

$$x = \frac{2t-1}{t-1} \;\Rightarrow\; xt - x = 2t - 1 \Rightarrow xt - 2t = x - 1 \Rightarrow t(x-2) = x - 1 \Rightarrow t = \frac{x-1}{x-2}$$
$$y = \frac{2t}{t-1} \;\Rightarrow\; yt - y = 2t \Rightarrow yt - 2t = y \Rightarrow t(y-2) = y \Rightarrow t = \frac{y}{y-2}$$

Again equate the two t's $\Rightarrow \frac{x-1}{x-2} = \frac{y}{y-2}$ and cross multiply to get

$$xy - 2y = xy - 2x - y + 2$$

Now cancel the xy's and rearrange to get the Cartesian form of $\boxed{2x - y - 2 = 0}$.

To get the obtuse angle between these two lines use $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

The slope, m_1, of $3x + y - 9 = 0$ is -3 and the slope, m_2, of $2x - y - 2 = 0$ is 2.

Therefore $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \tan\theta = \left|\frac{-5}{-5}\right| = \tan\theta = 1 \Rightarrow \theta = 45^\circ$.

This is obviously the acute angle so the obtuse angle is found by 180° - 45° = 135°.

LINEAR TRANSFORMATIONS (LINEAR COMBINATIONS)

To date we have met three types of transformations; central symmetry, axial symmetry and translations. They all have the same effect and that is they move points from one place to another.

Linear combinations also do just that. They move points from one place to another, however, they do so by using both the x and y value of the point being moved and substitute them into a function involving x and y.

For example $f(x, y) = (x', y')$ where $x' = 3x + 4y$ and $y' = 2x - 5y$
They are referred to as combinations because they combine the x and y coordinates of a single point into a given formula so that we can find the image point (x', y').

It is important to remember the three things these combinations do not conserve are perpendicularity, distance and area. Image lines must be tested for perpendicularity always.

Finding the image of line segments

Remember that all line segments have their equations written in the form $ax + by + c = 0$

Linear transformations or combinations will always be given with x' and y' written in terms of x and y. The easy way to find the image of a line segment through a linear combination is to rewrite x and y in terms of x' and y'. This is done using the following method:

1. Form a pair of simultaneous equations with the x' and y' formulae.
2. Eliminate the x's to find y in terms of x' and y'
3. Using the simultaneous equations again, eliminate the y's to find x in terms of x' and y'.(You may, of course, find x by the original substitution method!)
4. Substitute these new x and y values into the original equation of the line segment.
5. Rearrange the terms and form the equation of the image line. The resulting line will be written in terms of x' and y'.

Example:

F is the transformation $(x, y) \to (x', y')$, where $x' = px + qy, y' = rx + sy$.
If $f(1,2) = (8,-6)$ and $f(-3,-1) = (-9,-7)$, find the values of *p, q, r* and *s*.
L: $2x + 3y - 7 = 0$ and K: $3x - 2y - 4 = 0$
Verify that L $\perp$ K. Investigate if *f*(L) $\perp$ *f*(K)

Solution:

If $f(1,2) = (8,-6)$ then $x = 1, y = 2, x' = 8, y' = -6$
$\Rightarrow x' = 8 \Rightarrow 8 = px + qy \Rightarrow 8 = p + 2q$
and $y' = -6 \Rightarrow -6 = rx + sy \Rightarrow -6 = r + 2s$

If $f(-3,-1) = (-9,-7)$ then $x = -3, y = -1, x' = -9, y' = -7$
$\Rightarrow x' = -9 \Rightarrow -9 = px + qy \Rightarrow -9 = -3p - q$
and $y' = -7 \Rightarrow -7 = rx + sy \Rightarrow -7 = -3r - s$

To solve for *p, q, r* and *s,* use simultaneous equations between like equations:

$8 = p + 2q$
$-9 = -3p - q (\times 2)$

$8 = p + 2q$
$-18 = -6p - 2q$

$-10 = -5p$
$\Rightarrow 2 = p$
$\Rightarrow 3 = q$

$-6 = r + 2s$
$-7 = -3r - s (\times 2)$

$-6 = r + 2s$
$-14 = -6r - 2s$

$-20 = -5r$
$\Rightarrow 4 = r$
$\Rightarrow -5 = s$

To Verify that L ⊥ K, $m_1 \times m_2 = -1$

Slope of L = $\frac{-a}{b} \Rightarrow \frac{-2}{3}$ Slope of K = $\frac{-a}{b} \Rightarrow \frac{-3}{-2} \Rightarrow \frac{3}{2}$

$\frac{-2}{3} \times \frac{3}{2} = -1 \Rightarrow m_1 \times m_2 = -1$ ✓ The are ⊥.

Now to get $f(L)$ and $f(K)$, we need to find out x and y in terms of x' and y'.

$x' = 2x + 3y (\times 2)$
$y' = 4x - 5y$

$2x' = 4x + 6y$
$-y' = -4x + 5y$

$2x' - y' = 11y$
$\frac{1}{11}(2x' - y') = y$

$x' = 2x + 3y (\times 5)$
$y' = 4x - 5y (\times 3)$

$5x' = 10x + 15y$
$3y' = 12x - 15y$

$5x' + 3y' = 22x$
$\frac{1}{22}(5x' + 3y') = x$

Now substitute these new x and y values into L and K. *(x11 and x22 are included in the equations in superscript to show how the denominators are cleared)*

$f(L) =^{\times 11} 2\left(\frac{1}{22}(5x' + 3y')\right) +^{\times 11} 3\left(\frac{1}{11}(2x' - y')\right) - 7 = 0$
$\Rightarrow 5x' + 3y' + 6x' - 3y' - 7 = 0$
$\Rightarrow 11x' - 7 = 0$
$\Rightarrow m_1 = 0$

$$f(K) =^{\times 22} 3\left(\frac{1}{22}(5x' + 3y')\right) -^{\times 22} 2\left(\frac{1}{11}(2x' - y')\right) - 4 = 0$$
$$\Rightarrow 15x' + 9y' - 8x' + 4y' - 4 = 0$$
$$\Rightarrow 7x' + 13y' - 4 = 0$$
$$\Rightarrow m_2 = \frac{-7}{13}$$

Since $m_1 \times m_2 \neq -1$, the image lines are not $\perp$.