Problem Solving for Irish Second level Mathematicians

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Thursday 18th October 2007

Senior Level

Time allowed: **60 minutes**

Rules and Guidelines for Contestants

1. You are **not** allowed to use a calculator or any measuring device (e.g. ruler or protractor).
2. **Use a pencil to fill out the answer sheet**. If you make a mistake, you can erase the error and correct it.
3. Write your name clearly (in block capitals) in the space provided in the answer sheet.
4. You should have some extra sheets of your own paper (or a refill pad) for rough work while you are doing the questions.
5. When you have decided on your answer for a particular question, carefully mark your choice for that question on the answer sheet.
6. Do not make any other marks on the answer sheet other than to write your name and to mark your answers to the questions.
7. Some of the questions are quite difficult, and we do not expect that many people will have time to think about all of them in 60 minutes. You will probably do better if you concentrate on a few rather than trying to guess the answer to all of the questions.
   The questions at the beginning are easier than those at the end.
   The problems are meant to encourage you to think! Don't be in a rush to mark your answer to any of the questions - take your time, read the questions carefully and make sure you understand what is being asked before you start to figure out the answer.
8. **There is no pass/fail mark in PRISM.** Correct answers will score one point each; incorrect or omitted answers will score zero.

Good luck and thank you for participating in PRISM.
We hope you will enjoy the problems!

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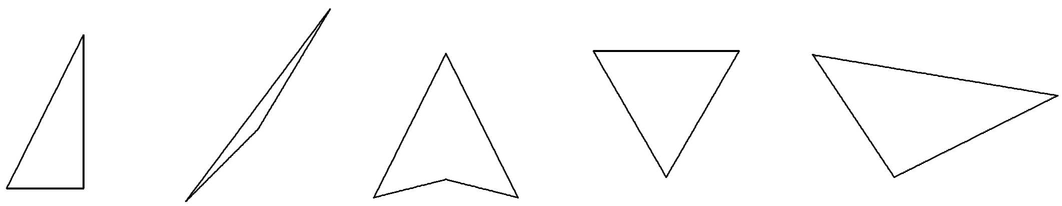
Senior Level 2007
SOLUTIONS

1. Which of the following is not a triangle?



(A) (B) (C) (D) (E)

Answer : C
Picture C shows a quadrilateral; the rest are triangles.

2. Which one of the following numbers is equal to $\frac{1}{3} - \frac{1}{12} - \frac{1}{4}$?

(A) $\frac{1}{7}$ (B) 0 (C) $\frac{1}{5}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

Answer : B
$\frac{1}{3} - \frac{1}{12} - \frac{1}{4} = \frac{4}{12} - \frac{1}{12} - \frac{3}{12} = \frac{4-1-3}{12} = \frac{0}{12} = 0.$

3. Which of the following is the number of centimetres in 10 kilometres?

(A) 10^5 (B) 10,000,000 (C) 1,000,000 (D) 10,000 (E) 10^3

Answer : C
There are 100 centimetres in a metre and 1000 metres in a kilometre. So the number of centimetres in one kilometre is $1000 \times 100 = 100,000$. Then the number of centimetres in *ten* kilometres is $10 \times 100,000 = 1,000,000$.

4. Which of the following numbers is the largest?

(A) $\sqrt{901}$ (B) $\frac{0.18}{0.006}$ (C) 2^5 (D) 3^3 (E) $(\sqrt{5})^4$

Answer : C
$\sqrt{901}$ is between 30 and 31 since $30^2 = 900$ and $31^2 = 961$.
$\frac{0.18}{0.006} = \frac{180}{6} = 30 < \sqrt{901}$
$2^5 = 32$
$3^3 = 27$
$\sqrt{5^4} = 5^2 = 25$

5. A square room has diagonal 10m in length. What is the area of the room?

(A) 10m^2 (B) 1m^2 (C) We need more information. (D) 100m^2 (E) 50m^2

Answer : E
Let the length in metres of a side of the square be a. Then by considering the right-angled triangle formed on the floor by two adjacent walls and a diagonal we can observe

$$a^2 + a^2 = 10^2 = 100.$$

So $2a^2 = 100$ and $a^2 = 50$. But a^2 is also the area in square metres of the room.

6. What is the value of $\sqrt{2^4 + \sqrt{3^4}}$?

(A) 5 (B) $\sqrt{97}$ (C) 25 (D) $\sqrt{13}$ (E) 7

Answer : A
$\sqrt{2^4 + \sqrt{3^4}} = \sqrt{16 + \sqrt{81}} = \sqrt{16 + 9} = \sqrt{25} = 5.$

7. Tap A can fill a tank in 20 minutes, Tap B can fill the tank in 12 minutes, and Tap C can fill the tank in 5 minutes. How long will it take for the three taps together to fill the tank?

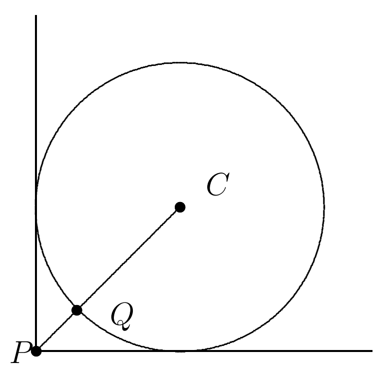
(A) 4 minutes (B) 3 minutes (C) 2 minutes (D) 37 minutes (E) 5 minutes

Answer : B
In one minute, Tap A will fill $\frac{1}{20}$ of the tank, Tap B will fill $\frac{1}{12}$ of the tank and Tap C will fill $\frac{1}{5}$ of the tank. Therefore the three taps working together will fill

$$\frac{1}{20} + \frac{1}{12} + \frac{1}{5} = \frac{3 + 5 + 12}{60} = \frac{20}{60} = \frac{1}{3}$$

of the tank in one minute. So it will take three minutes for the three taps working together to fill the tank.

8. In the diagram, the horizontal and vertical lines are perpendicular to each other and intersect at P. The circle just touches both lines, its centre is at C and its radius is 1. The line CP intersects the circle at Q as in the diagram. What is the distance from Q to P?



(A) $1 - \sqrt{2}$ (B) $\sqrt{2}$ (C) $\sqrt{2} - 1$ (D) 1 (E) $1 + \sqrt{2}$

Answer : C
By the Theorem of Pythagoras the length of the line segment $[CP]$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$. The length of the line segment $[CQ]$ is 1 since it is a radius of the circle. Hence the distance from Q to P is $\sqrt{2} - 1$.

9. On the first day of a chess tournament with five players, every player plays two games. How many games are played on that day?

   (A) 20 (B) 10 (C) 4 (D) 6 (E) 5

   Answer : E
   Every player plays two games, so there are 10 instances of a player playing a game. Since each game has two players, this accounts for 5 games.

10. In the latest series of *XY-Factor*, contestants competed against each other in solving quadratic equations. In the first programme one quarter of the contestants were eliminated. In the second programme one third of the remaining contestants were eliminated. In the third programme one half of the remaining contestants were eliminated. In the fourth and final programme, five contestants were eliminated leaving one winner of *XY-Factor*. How many contestants took part in the first programme?

    (A) 24 (B) 28 (C) 30 (D) 32 (E) 40

    Answer : A
    Since the fourth programme involved one winner and five eliminated players, it involved 6 players. These 6 accounted for half of the players in the third programme, so the third programme had 12 players. These 12 accounted for two-thirds of the players in the second programme, so the second programme had 18 players. These 18 accounted for three-quarters of the players in the first programme, so the first programme had 24 players.

11. The average of a set of ten numbers is 10. If one of the numbers is removed, the average of the remaining 9 numbers is 9. What number was removed?

    (A) 11 (B) 19 (C) 9 (D) 20 (E) 10

    Answer : B
    If the average of ten numbers is 10, the sum of these ten numbers is $10 \times 10 = 100$. If the average of nine numbers is 9, the sum of these nine numbers is $9 \times 9 = 81$. So removing one number reduced the sum from 100 to 81 - therefore the number that was removed is $100 - 81 = 19$.

12. You are asked to paint each face of a cube with a colour, in such a way that two faces that share a common edge must not have the same colour. What is the minimum number of colours that you will need?

    (A) 2 (B) 6 (C) 4 (D) 5 (E) 3

    Answer : E
    A pair of faces can be given the same colour only if they are opposite, since two non-opposite faces on a cube always have an edge in common. Since there are three pairs of opposite faces, at least three colours are needed. On the other hand three colours are enough - we can colour the top and bottom faces red, the front and back faces blue and the left and right faces black. Then no faces that share a common edge have the same colour.

13. In the sport of prism-ball, five points are awarded for a goal and seven points are awarded for a touchdown. The only ways to score points are through goals and touchdowns. In last week's game, Team A scored a total of 29 points. How many touchdowns did they score?

    (A) 3 (B) 4 (C) 0 (D) 2 (E) 1

    Answer : D
    The possible numbers of points scored by Team A for touchdowns are the multiples of 7 that are less than (or equal to) 29. These are 0, 7, 14, 21 and 28. The remaining points were scored through goals, so the number of remaining points must be a multiple of five. Of the available possibilities only 14 leaves a multiple of five when subtracted from 29, so Team A scored 14 points through touchdowns and 15 through goals. They scored 2 touchdowns.

14. How many numbers between 100 and 999 have the sum of their three digits equal to 10?

(A) 54 (B) 100 (C) 63 (D) 46 (E) 55

Answer : A

First suppose that the first digit is 1. Then the second and third digits must add up to 9. We have ten choices for the second digit, each of which leads to only one choice for the third digit. For example choosing the second digit to be 0 forces the third to be 9, choosing the second to be 1 forces the third to be 8 etc. So there are 10 possibilities with 1 as the first digit.

Now suppose the first digit is 2. Then the second and third must add up to 8. We can't choose the second to be 9 or we have already exceeded 10 as the sum of the first two digits. However any of the other 9 possibilities for the second digit will do, so there are 9 possibilities with the first digit equal to 2.

If the first digit is 3, the second cannot be 8 or 9, but any of the other 8 possiblilities are permissible and each immediately determines the third digit. So there are 8 possibilities with the first digit equal to 2.

Continuing in this way, we find 7 possibilities with the first digit equal to 3, etc. Finally if the first digit is 9, the second can only be 0 or 1, and there are two possibilies (901 and 910).

Thus the total number of numbers in the range 100 to 999 in which the sum of the digits is 10 is

$$10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 54.$$

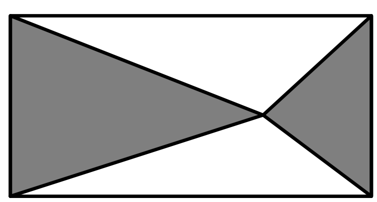
15. Which of the following is an integer?

(A) 7π (B) $\dfrac{1}{1+\sqrt{2}} + \dfrac{1}{1-\sqrt{2}}$ (C) $\sqrt{10^{20}+1}$ (D) $\dfrac{2^{50}}{3}$ (E) $\sqrt{1000}$

Answer : B

$$\frac{1}{1+\sqrt{2}} + \frac{1}{1-\sqrt{2}} = \frac{(1-\sqrt{2})+(1+\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{2}{1-2} = -2.$$

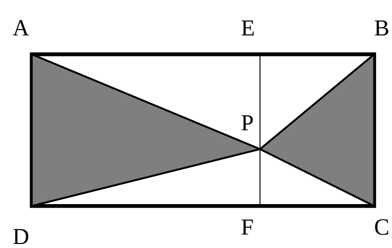
16. In the diagram below, the rectangle has area 40. What is the total area of the shaded regions?



(A) 10 (B) 20 (C) 15 (D) More information is needed (E) 40

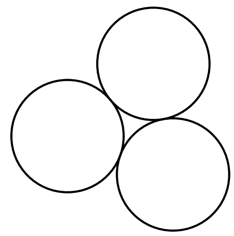
Answer : B

Label the points as in the diagram below.



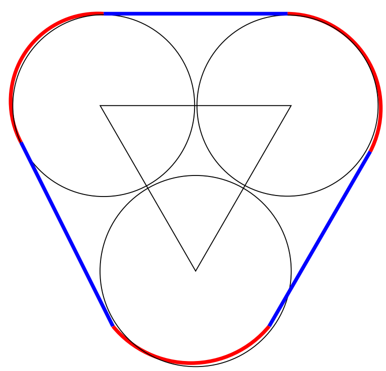
By considering the side AD as the base of the triangle APD, we can see that this triangle accounts for half of the area of the rectangle $AEFD$. Similarly the triangle BCP accounts for half of the area of the rectangle $BEFC$. Thus the total are of the shaded regions is half of the total area of the rectangle $ABCD$.

17. Three cylindrical barrels each having a diameter of 1 meter and a height of 1 metre are standing side by side on level ground. All touch each other as in the diagram below (which shows the view from above). Find the length in metres of the shortest band that can be tied around the outside of the three barrels.



(A) $\frac{3}{2}\pi$ (B) $3+\pi$ (C) 6 (D) π (E) $6+2\pi$

Answer : B



In the diagram the vertices of the triangle are the centres of the three circles, and the shortest band consists of the three red arcs and the three blue line segments. The length of each blue line segment is the distance between the centres of two different circles which is 1 metre, and each red arc accounts for one-third of a circle. Since each circle has circumference π, each red arc has length $\pi/3$. Thus the total length in metres of the band is $3+3(\frac{\pi}{3})=3+\pi$.

18. Conor and Orla are training for a long distance cycling race. Last Saturday morning they started cycling at the same time, Orla from Clifden to Galway and Conor from Galway to Clifden. They both used the same route, and each travelled at a constant speed throughout the journey. Orla reached Galway at 4:00 in the afternoon, and Conor reached Clifden at 2:15 in the afternoon. They met on the road at 12:00. At what time did they start cycling?

(A) 9:00 (B) 9:30 (C) 8:30 (D) 8:00 (E) More information is needed.

Answer : A
Let t be the time at which they start cycling. Call the section of the route between Galway and the meeting place Section A of the route, and call the section between the meeting point and Clifden Section B. So Conor spends 2.25 hours (2hrs 15 mins) on Section A and $12-t$ hours on Section B. Orla spends $12-t$ hours on Section A and 4 hours on Section B. Since both cyclists travel at a constant rate, the ratio of time spent on Section A to time spent on Section B must be the same for both of them. Thus

$$\frac{12-t}{4}=\frac{2.25}{12-t}\Longrightarrow(12-t)^2=4\times 2.25=9.$$

Thus $12-t=3$ (since $12-t$ must be positive). So $t=9$, and they started cycling at 9:00.

19. Every person in Ballylogic is either a truth-teller or a liar; truth-tellers always tell the truth and liars always lie. One day in Ballylogic, Ann says "Bob is a truth-teller". Then Bob says "Ann, Cathy and I are . . . all truth-tellers" but at that moment there is a loud noise and you are not sure if he said "Ann, Cathy and I are all truth-tellers" or "Ann, Cathy and I are not all truth-tellers". Then Cathy (who heard what Bob said) says "Bob said that we are not all truth-tellers". How many of the three are truth-tellers?

(A) 1 (B) 2 (C) none (D) 3 (E) More information is needed

Answer : C

(a) Suppose that Ann is a truth-teller. Then Bob is a truth-teller.

i. If Bob said "We are all truth-tellers" then they are all truth-tellers - but this is impossible since in this case Cathy lies about what Bob said.

ii. If Bob said "We are not all truth-tellers" then since he and Ann *are* truth tellers, it must be that Cathy is a liar. But Cathy is here telling the truth about Bob.

So the assumption that Ann is a truth teller cannot lead to a sensible conclusion.

(b) Thus Ann is a liar, and since Ann must be lying when she speaks about Bob, Bob is a liar too. So Bob must have said "We are all truth-tellers" since he must lie. Then Cathy lies about what Bob said, so she is a liar too.

The conclusion is that Ann, Bob and Cathy are all liars.

20. How many integers n are there such that $n + 20$ and $n - 20$ are both perfect squares? (A *perfect square* is a number that is the square of an integer).

(A) 4 (B) 2 (C) 1 (D) None (E) Infinitely many

Answer : B

Suppose that n is an integer such that $n+20 = a^2$ and $n-20 = b^2$ where a and b are both positive integers and $a > b$. It follows that

$$a^2 - b^2 = (n + 20) - (n - 20) = 40.$$

Thus $(a - b)(a + b) = 40$. Note also that $(a - b)$ and $(a + b)$ must be both even or both odd, since their difference $2b$ is an even integer. In fact then $(a-b)$ and $(a+b)$ must be both even, since their product is 40, an even number. So the possible values of $a - b$ and $a + b$ depend on the number of ways of writing 40 as the product of two even integers. There are two ways to do this, namely $40 = 2 \times 20$ and $40 = 4 \times 10$.

Case 1: Suppose $a - b = 2$ and $a + b = 20$. Then $2a = (a - b) + (a + b) = 22$, so $a = 11$ and $b = 9$. Then since $n + 20 = a^2$ we have $n = 121 - 20 = 101$. (Of course $101 - 20 = 81$ and $101 + 20 = 121$, so in this case $n-20$ and $n+20$ are both squares). *Case 2*: Suppose $a-b = 4$ and $a+b = 10$. Then $2a = (a - b) + (a + b) = 14$, so $a = 7$ and $b = 3$. Then since $n + 20 = a^2$ we have $n = 49 - 20 = 29$. Again $n - 20 = 9$ and $n + 20 = 49$ are both squares.

So the possible values of n are 29 and 101.

We hope that you enjoyed participating in PRISM2007 and that you might enjoy thinking more about these problems. If you have any questions or comments about the problems or solutions, please send them to me at `rachel.quinlan@nuigalway.ie`.