Problem Solving for Irish Second level Mathematicians
Thursday 18th October 2007

Junior Level

Time allowed: 60 minutes

Rules and Guidelines for Contestants

1. You are **not** allowed to use a calculator or any measuring device (e.g. ruler or protractor).
2. **Use a pencil to fill out the answer sheet**. If you make a mistake, you can erase the error and correct it.
3. Write your name clearly (in block capitals) in the space provided in the answer sheet.
4. You should have some extra sheets of your own paper (or a refill pad) for rough work while you are doing the questions.
5. When you have decided on your answer for a particular question, carefully mark your choice for that question on the answer sheet.
6. Do not make any other marks on the answer sheet other than to write your name and to mark your answers to the questions.
7. Some of the questions are quite difficult, and we do not expect that many people will have time to think about all of them in 60 minutes. You will probably do better if you concentrate on a few rather than trying to guess the answer to all of the questions.
   The questions at the beginning are easier than those at the end.
   The problems are meant to encourage you to think! Don't be in a rush to mark your answer to any of the questions - take your time, read the questions carefully and make sure you understand what is being asked before you start to figure out the answer.
8. **There is no pass/fail mark in PRISM.** Correct answers will score one point each; incorrect or omitted answers will score zero.

Good luck and thank you for participating in PRISM.
We hope you will enjoy the problems!

PRISM

Problem Solving for Irish Second level Mathematicians

Junior Level 2007

1. Which of the following numbers is not equal to 6?

   (A) The number of faces in a cube.
   (B) The number of sides in a hexagon.
   (C) The number of buttons on a typical mobile phone.
   (D) The number of teams in the Six Nations Rugby tournament.
   (E) $3 + 3$

   Answer: C.

2. Which of the following is equal to $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$?

   (A) $\frac{13}{12}$ **(B)** $\frac{3}{9}$ **(C)** $\frac{3}{12}$ **(D)** $\frac{1}{12}$ **(E)** 1

   Answer: A. We have

$$\begin{aligned}\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} &= \tfrac{6}{12} + \tfrac{4}{12} + \tfrac{3}{12} \\ &= \tfrac{13}{12}\end{aligned}$$

3. Which of the following is equal to $\dfrac{0.3}{0.06}$?

   (A) 2 **(B)** 50 **(C)** 0.5 **(D)** 3 **(E)** 5

   Answer: E. Multiply above and below by 100 to make it obvious.

4. Which of the following numbers is the largest?

   (A) $2 \times 3 \times 4$ **(B)** 5^2 **(C)** $\frac{50}{3}$ **(D)** $2 + 3 + 4 + 5$ **(E)** 20

   Answer: B. Clearly $25 = \frac{50}{2} > \frac{50}{3}$.

5. You have one hour to complete the PRISM contest. After 35 minutes what fraction of your time remains?

   (A) $\frac{1}{2}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{4}$

   Answer: B. You have 25 minutes left. So the fraction of your time that remains is $\frac{25}{60} = \frac{5}{12}$.

6. Tom is five years younger than Mary. Two years from now, Mary will be twice as old as Tom. What age is Tom now?

(A) 5 **(B)** 8 **(C)** 10 **(D)** 3 **(E)** 4

Answer: D. Suppose that Tom is n years old. Then Mary is $n+5$ years old and we are told that

$$(n+5)+2 = 2(n+2)$$

Therefore $n = 3$.

7. Two apples and three oranges cost €2.80. Three apples and four oranges cost €3.90. How much do two apples and two oranges cost?

(A) €2.20 **(B)** €1.00 **(C)** €1.10 **(D)** €2.10 **(E)** €2.50

Answer: A. One apple and one orange must cost €1.10 - the difference between the cost of two apples and two oranges and the cost of three apples and four oranges. Therefore two apple and two oranges will cost €2.20.

8. Which of the following is the number of seconds in a week?

(A) $60 \times 60 \times 24 \times 7$

(B) 3600×7

(C) 500000

(D) $60 + 60 + 24 + 7$

(E) $60 \times 60 \times 48 \times 3$

Answer: A. There 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day and 7 days in a week. Therefore the total number of seconds in a week is $60 \times 60 \times 24 \times 7$.

9. What is the 2007th letter of this sequence?

P,R,I,S,M,P,R,I,S,M,P,R,I,S,M,P,R,...

(A) P **(B)** R **(C)** I **(D)** S **(E)** M

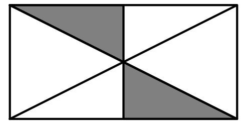
Answer: B. The sequence repeats every 5 letters. So the 5th, 10th, 15th, 20th, 25th , ...2000th and 2005th letter are all M. That means that the 2005th letter is M, the 2006th letter is P and the 2007th letter is R.

10. Tap A can fill a certain swimming pool in 40 minutes. Tap B can fill the same pool in 20 minutes and tap C can fill the pool in 8 minutes. How many minutes will it take the three taps together to fill the pool?

(A) 5 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 20

Answer: A. In 40 minutes the three taps together will fill the swimming pool $1+2+5 = 8$ times over. So it will take them 5 minutes to fill the pool once.

11. The large rectangle shown below has sides of length 1 and 2. The two shaded triangles are right-angled triangles. What is the total area of the shaded region?



(A) 1 **(B)** 0.25 **(C)** 0.5 **(D)** $\frac{11}{10}$ **(E)** $\frac{2}{\sqrt{17}}$

Answer: C. By symmetry the total shaded area is $\frac{1}{4}$ of the total area of the big rectangle. The big rectangle has area 2.

12. There are 150 houses in a new housing estate. A carpenter is putting numbers on all the doors. Each digit requires 1 screw to attach, and the houses are to be numbered with the numbers 1 to 150. How many screws will the carpenter need in order to complete the task?

(A) 150 **(B)** 450 **(C)** 351 **(D)** 300 **(E)** 342

Answer: E. 150 houses have at least one digit, (150-9) houses have at least 2 digits and (150 - 99) houses have 3 digits. Therefore, the total number of screws that the carpenter needs is $150 + (150 - 9) + (150 - 99) = 450 - 9 - 99 = 342$.

13. A gambler goes into a casino and doubles his money. He leaves and pays €8 for his parking. He goes to another casino, doubles his money and leaves. He pays another €8 for his parking and finds that he has no money left. How much money did he have before he entered the first casino?

(A) €8 **(B)** €6 **(C)** €4 **(D)** €10 **(E)** €0

Answer: B. Let x be the amount in euro that the gambler has before. Then

$$2(2x - 8) - 8 = 0$$

Therefore $x = 6$.

14. Which of the following numbers is the largest?

(A) $\sqrt{5}$ **(B)** $\sqrt[3]{11}$ **(C)** 2 **(D)** $\frac{11}{5}$ **(E)** 2.1

Answer: A. Observe that $(\sqrt[3]{11})^2 = \sqrt[3]{11^2} = \sqrt[3]{121} < \sqrt[3]{125} = 5$. Therefore $\sqrt[3]{11} < \sqrt{5}$. Similarly, $(\frac{11}{5})^2 = \frac{121}{25} < \frac{125}{25} = 5$, so $\frac{11}{5} < \sqrt{5}$. Also $2 < 2.1 < \frac{11}{5}$.

15. A set of ten numbers has an average of 10. If one of the numbers is removed then the average of the remaining nine numbers is 9. The removed number is

(A) 9 **(B)** 10 **(C)** 11 **(D)** 19 **(E)** 20

Answer: D. The sum of all ten of the original numbers must be 100. The sum of the nine remaining numbers must be 81. Therefore, the number removed was $(100 - 81) = 19$.

16. A list of ten numbers contains two each of the numbers 0, 1, 2, 3 and 4. The two 0s are next to each other, the two 1s are separated by one number, the two 2s are separated by two numbers, the two 3s are separated by three numbers and the two 4s are separated by four numbers. The list starts $3, 4, \ldots$. What is the last number in the list?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer: B. Suppose that the sequence is

$$3, 4, c, d, e, f, g, h, i, j$$

where c, d, e, f, g, h, i, j are to be determined. We know, from the given information, that $e = 3$ and $g = 4$. So the sequence is

$$3, 4, c, d, 3, f, 4, h, i, j$$

Now c must be 0, 1 or 2. c cannot be 1, since that would imply that $e = 1$ also, but we know that $e = 3$. If $c = 2$ then $f = 2$ also. But this would mean that d must be 0, 1 or 2 but this is impossible. So it must be that $c = 0$. Therefore our sequence is

$$3, 4, 0, 0, 3, -, 4, -, -, -$$

where the four remaining blanks consist of two 1s and two 2s. It is easy to see that the only possibility is

$$3, 4, 0, 0, 3, 2, 4, 1, 2, 1$$

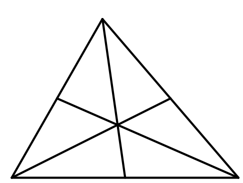
17. If $ab = 10$ and $a + b = 20$, what is $\dfrac{1}{a} + \dfrac{1}{b}$?

(A) $\frac{1}{2}$ **(B)** $\frac{4}{5}$ **(C)** 1 **(D)** 2 **(E)** 3

Answer: D. Observe that

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{b+a}{ab} \\ &= \frac{20}{10} \\ &= 2 \end{aligned}$$

18. How many triangles are visible in the diagram below? (A triangle is visible if all its edges are drawn in black in the diagram.)



(A) 20 **(B)** 16 **(C)** 7 **(D)** 6 **(E)** 15

Answer: B. This is an exercise in careful counting. Consider an edge of the large triangle. Apart from the large triangle, there are 3 other visible triangles that contain that edge. Now, if a visible triangle does not contain one of the edges of the large triangle, then it must have one of its vertices at the centre point, It is easy to see that there are 6 such triangles - the six smallest triangles that are clearly visible in the diagram. Thus there are a total of

$$1 + 9 + 6 = 16$$

visible triangles.

19. There are 6 players in a chess tournament. Each player plays each of the other players once. There are no draws. Exactly one of the following statements cannot be false. Which one is it?

(A) At least two players win at least 3 games each.

(B) Some player wins 5 games.

(C) There is a player who wins 4 games and loses 1 game.

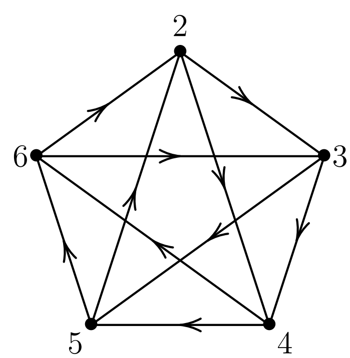
(D) There is a player who loses at least 3 games.

(E) At least one player loses 5 games.

Answer: D. Let us call the players 1, 2, 3, 4, 5 and 6.

Consider statement (B). This can be false, since we could have 1 beats 2, 2 beats 3, 3 beats 4, 4 beats 5, 5 beats 6, and 6 beats 1. Now, no matter what the results of the other games, each player has lost a game and thus (B) is false. The same configuration also shows that statement (E) can be false.

For statements (A) and (C) consider the following set of results: Suppose that 1 wins all five games, 2 beats 3, 3 beats 4, 4 beats 5, 5 beats 6, and 6 beats 2. Also suppose that 2 beats 4, 4 beats 6, 6 beats 3, 3 beats 5, and 5 beats 2. Then player 1 has won 5 and lost 0, and all the other players have won 2 and lost 3. Thus both (A) and (C) could be false. The following picture shows the symmetry in the results among the players 2, 3, 4, 5 and 6 in this configuration. An arrow pointing from i to j means that i beat j.



Finally, we can show that (D) must be true. Suppose that it is false - that is, no player loses more than 2 games. There are 6 players, which means that there are at most $6 \times 2 = 12$ defeats in the tournament. But there are a total 15 games played in the tournament, each of which must have a loser! Therefore (D) must be true.

20. What is the remainder when

$$1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 + 19^2 + 21^2 + 23^2 + 25^2 + 27^2 + 29^2 + 31^2$$

is divided by 4?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer: A. Note that each term of the sum is $(2k+1)^2$ for some integer k. Now $(2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, so the remainder of each term on division by 4 is 1. Notice that there are 16 terms in the sum. Therefore the required remainder is the same as the remainder when dividing 16 by 4, which is 0.

Commentary: The first five problems are meant to test basic arithmetic skills. Problems 6 through 18 involve some extra mathematical skills such as equation solving, modelling, counting and some deductive logic, in varying degrees. Problem 18 seems rather simple but it is a nice exercise in careful counting. Unless the student is very careful, it is easy to double count or to miss some of the triangles. Of course the last two problems are harder than the others, but they can still be solved without recourse to any advanced mathematics - you just need to be willing to spot the patterns. For problem 20, it is possible to observe the fact that the square of an odd integer has remainder 1 on division by 4, even if the student cannot prove this. This should lead her to guess the correct answer.

As an aside, problem 19 was inspired by the rugby world cup. There was much discussion at the time of the relative merits of Ireland winning 3 triple crowns in 4 years. This made me wonder whether or not it is possible to win the triple crown and still finish last in the championship. The solution to problem 19 shows that it is not - which is good for Irish rugby, I think! Of course if the 6 nations ever becomes the 7 nations (say Argentina join) then it will be possible to win the triple crown and the wooden spoon in the same year.

I hope that you enjoyed these problems and that they might inspire some classroom discussion. If you have any comments, I'd like to hear them. Email: james.cruickshank@nuigalway.ie